PHYSICAL REVIEW E **77**, 030102(R) (2008)

Absorbing-state phase transitions: Exact solutions of small systems

Ronald Dickman*

Departamento de Física, ICEx, Universidade Federal de Minas Gerais, 30123-970, Belo Horizonte, Minas Gerais, Brazil

(Received 23 September 2007; revised manuscript received 9 February 2008; published 6 March 2008)

I derive precise results for absorbing-state phase transitions using exact (numerically determined) quasistationary (QS) probability distributions for small systems. Analysis of the contact process on rings of 23 or fewer sites yields critical properties (control parameter, order-parameter ratios, and critical exponents z and $\beta/\nu_\perp$) with an accuracy of 0.06% or better; precise results are also obtained for the pair contact process. The QS distribution yields insights on the statistical entropy of these models. Preliminary application to a model in the stochastic sandpile class is also described.

DOI: 10.1103/PhysRevE.77.030102 PACS number(s): 05.50.+q, 02.50.Ga, 05.40.−a, 05.70.Ln

Stochastic processes with an absorbing state arise frequently in statistical physics [1,2]. In systems with spatial structure, phase transitions to an absorbing state, as exemplified by the contact process [3,4], are widely studied in connection with self-organized criticality [5], the transition to turbulence [6], and issues of universality in nonequilibrium critical phenomena [7,10–12]. Interest in such transitions should continue to grow in the wake of experimental confirmation in a liquid crystal system [13]. This paper presents an approach to absorbing-state phase transitions via analysis of exact (numerical) quasistationary (QS) probability distributions.

The quasistationary probability distribution (QSD) provides a wealth of information about systems exhibiting an absorbing-state phase transition [14,15]. (Since the only true stationary state for a finite system is the absorbing one, "stationary-state" simulations in fact probe QS properties, that is, conditioned on survival.) In particular, the order parameter and its moments, static correlation functions, and the QS lifetime are all accessible from the QSD. Until now, QS properties of systems with spatial structure have been determined only via simulation [15–17]; here I develop an effective scheme for determining the QSD on rings of L sites.

The QSD is defined as follows. Consider a continuous-time Markov process X_t with state A absorbing: if $X_t=A$ then $X_{t'}=A$ at all subsequent times. The transition rates $w_{C',C}$ (from state C to state C') are such that $w_{C,A}=0$, $\forall$ C. (Some processes have several absorbing states, $A_1,\ldots,A_n$.) Let $p_C(t)$ denote the probability of state C at time t, given some initial state $X_0\neq A$. The survival probability $P_s(t)=\Sigma_{C\neq A}p_C(t)$ is the probability that the process has not visited the absorbing state up to time t. We suppose that as $t\to\infty$ the p_C, normalized by the survival probability, attain a time-independent form, allowing us to define the QSD:

$$\bar{p}_C=\lim_{t\to\infty}\frac{p_C(t)}{P_s(t)}\quad (C\neq A), \tag{1}$$

with $\bar{p}_A\equiv 0$; it is normalized so $\Sigma_{C\neq A}\bar{p}_C=1$.

Since numerical integration of the master equation for processes with a large state space is very time-consuming, I use instead the iterative scheme demonstrated in Ref. [18]. Given some initial guess for the distribution $\bar{p}_C$, the following relation is iterated until convergence is achieved:

$$\bar{p}'_C=a\bar{p}_C+(1-a)\frac{r_C}{w_C-r_A}. \tag{2}$$

Here $r_C=\Sigma_{C'}w_{C,C'}\bar{p}_{C'}$ is the probability flux (in the master equation) into state C (r_A is the flux to the absorbing state, so that $1/r_A$ gives the lifetime of the QS state), and $w_C=\Sigma_{C'}w_{C',C}$ is the total rate of transitions out of state C. The parameter a can take any value between 0 and 1; in practice rapid convergence is obtained with $a=0.1$.

The iterative scheme is used to determine the QSD of the contact process (CP) on rings of L sites. In the CP [3,4,7], each site i of a lattice is either occupied $[\sigma_i(t)=1]$, or vacant $[\sigma_i(t)=0]$. Transitions from $\sigma_i=1$ to $\sigma_i=0$ occur at a rate of unity, independent of the neighboring sites. The reverse transition is only possible if at least one neighbor is occupied: the transition from $\sigma_i=0$ to $\sigma_i=1$ occurs at rate λr, where r is the fraction of nearest neighbors of site i that are occupied; thus the state $\sigma_i=0$ for all i is absorbing. (λ is a control parameter governing the rate of spread of activity.)

Although no exact results are available, the CP has been studied intensively via series expansion, Monte Carlo simulation, exact diagonalization [8], and real-space renormalization group analysis [9]. Since its scaling properties have been discussed extensively [7,10,11] I review them only briefly here. The best estimate for the critical point in one dimension is $\lambda_c=3.29785(2)$, as determined via series analysis [19]. Approaching the critical point, the correlation length ξ and lifetime τ diverge, following $\xi\propto|\Delta|^{-\nu_\perp}$ and $\tau\propto|\Delta|^{-\nu_\parallel}$, where $\Delta=(\lambda-\lambda_c)/\lambda_c$ is the relative distance from the critical point. The order parameter (the fraction of active sites) scales as $\rho\propto\Delta^\beta$ for $\Delta>0$. Near the critical point, finite-size scaling (FSS) [20,21] implies that average properties such as ρ depend on L through the scaling variable $\Delta L^{1/\nu_\perp}$, leading, at the critical point, to $\tau\propto L^z$, with dynamic exponent $z=\nu_\parallel/\nu_\perp$, and $\rho\propto L^{-\beta/\nu_\perp}$.

The computational algorithm for determining the QSD consists of three components. The first enumerates all configurations on a ring of L sites. Configurations differing only by a lattice translation are treated as equivalent. In subsequent stages only one representative of each equivalence

*dickman@fisica.ufmg.br

 ©2008 The American Physical Society

class is used, yielding a considerable speedup and reduction in memory requirements. (The multiplicity, or number of configurations associated with each equivalence class, is needed for calculating observables.) The second component runs through the list of configurations, enumerating all possible transitions. Here proper care must be taken to determine the weight of each transition, due to the varying multiplicity of initial and final configurations. The exit rate for each configuration C is simply $w_C = n_C + (\lambda/2)c_C$, where n_C is the number of occupied sites and c_C the number of occupied-vacant nearest-neighbor pairs. To determine the flux r_C one enumerates all transitions into C from other configurations C'. (Each vacant site i in C implies a transition from a configuration C', differing from C only in that site i is occupied; each nearest-neighbor pair of occupied sites $i, i+1$ in C implies transitions from configurations C' in which either i or $i+1$ is vacant. Transitions between the same pair of configurations C' and C are grouped together, with the proper multiplicity stored in the associated weight.) The final part of the algorithm determines the QSD via the iterative procedure described above. (For larger sizes, enumeration of configurations and transitions is time consuming, and it is advantageous to store these data in an external file.) The specific rules of the model enter only in the second stage; extension to other models is straightforward.

I determined the QSD for the contact process on rings of up to 23 sites. The number of configurations scales as $N_c \simeq (2^L - 2)/L + 1$ (for L prime this formula is exact). The number of annihilation transitions is $N_a \simeq 2^{L-1}$ (on average half the sites are occupied) and that of creation transitions is $N_{\rm cr} \simeq 3 \times 2^{L-3}$. (For $L=23$, there are 364723 configurations, and $\approx 7.3 \times 10^6$ transitions; the calculation takes about 16 hours on a 3 GHz processor.)

The QS order parameter, given by $\rho = L^{-1} \Sigma_C \bar{p}_C n_C$, with n_C the number of occupied sites in configuration C, follows the expected trend [i.e., $\rho(\lambda)$ is a sigmoidal function that becomes sharper with increasing L], but does not show, for these small sizes, a clear sign of the critical point (see Fig. 1). To determine critical properties I apply finite-size scaling analysis [21]. Since $1/r_A$ is the lifetime of the QS state, one has $r_A(\Delta, L) \simeq L^{-z} \mathcal{F}(\Delta L^{1/\nu_\perp})$, where $\mathcal{F}$ is a scaling function. (In fact, the present results for the QS lifetime reproduce those reported in Ref. [8].) The quantities

$$R_L(\lambda) \equiv \frac{\ln[r_A(\lambda, L-1)/r_A(\lambda, L)]}{\ln[L/(L-1)]} \tag{3}$$

are therefore expected to converge to z as $L \to \infty$. The condition $R_{L+1}(\lambda_{c,L}) = R_L(\lambda_{c,L}) = z_L$ then defines a sequence of estimates for λ_c and the exponent z. Using high-resolution studies, with $\Delta\lambda = 2 \times 10^{-5}$ in the vicinity of $\lambda_{c,L}$, precise estimates of $\lambda_{c,L}$ and z_L (uncertainty $\sim 10^{-13}$) are obtained via Neville's algorithm [22]. Applying Bulirsch-Stoer (BST) extrapolation [23] to the data for $\lambda_{c,L}$, I find $\lambda_c = 3.29791(1)$ and $z = 1.5708(3)$. (These results are essentially the same as reported in Ref. [8]; as in that reference, the parameter ω of the BST procedure is adjusted so as to minimize the difference between the penultimate entries of the BST table. Uncertainties are estimated from the difference between the result obtained using all the data, and that for $L \leqslant 22$.) The result for λ_c is about 0.002% above the series value quoted above; for comparisons of critical exponents with literature values, see Table I. In a similar manner, one can calculate the exponent ratio $\beta/\nu_\perp$ from crossings of the quantities

$$S_L(\lambda) \equiv \frac{\ln[\rho(\lambda, L)/\rho(\lambda, L-1)]}{\ln[L/(L-1)]}. \tag{4}$$

BST extrapolation yields $\beta/\nu_\perp = 0.25218(2)$.

Critical behavior at an absorbing state phase transition is also characterized by order-parameter moment ratios [24]. Let m_j denote the jth moment of the occupied site density and $r_{211} \equiv m_2/m_1^2$. BST extrapolation of the data for $L=8$ to 23 at λ_c furnish $r_{211,c} = 1.17370(5)$. [The values $\lambda_{r,L}$, marking the crossing of $r_{211}(L)$ and $r_{211}(L+1)$, are expected to approach the critical value systematically. Extrapolation of the

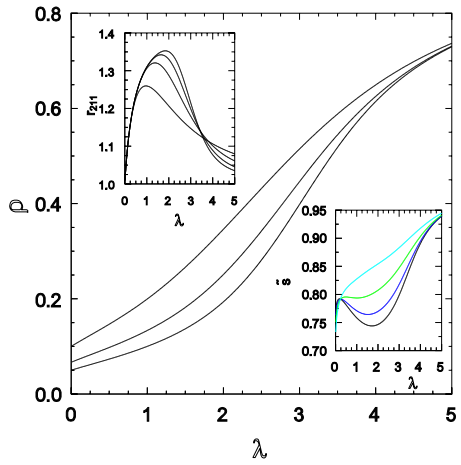

FIG. 1. (Color online) QS order parameter vs creation rate λ in the CP; system sizes $L=10$, 15, and 20 (upper to lower). Upper inset: moment ratio r_{211} for system sizes 5, 10, 15, and 20, in order of increasing maximum value. Lower inset: reduced entropy per site $\tilde{s}$ for sizes 12, 16, 20, and 23 (upper to lower).

TABLE I. Critical parameters for the CP and PCP (present study) compared with literature values for directed percolation (DP), from Refs. [26,24].

	z	$\nu_\perp$	$\beta/\nu_\perp$	r_{211}	q
CP	1.5808(3)	1.0962(7)	0.25198(2)	1.17370(5)	−0.5105(5)
PCP	1.574(1)	1.094(1)	0.252(4)	1.1747(5)	−0.49(2)
DP	1.5807(1)	1.096854(4)	0.25208(5)	1.1736(1)	−0.503(3)

crossing values yields the somewhat less precise estimate $\lambda_c=3.2961(15)$.] The moment ratio provides an efficient method for calculating the exponent $\nu_\perp$. From the finite-size scaling form $r_{211}=\mathcal{R}(\Delta L^{1/\nu_\perp})$, one has $r'_L \equiv dr_{211,L}/d\lambda|_{\lambda_c} \sim L^{1/\nu_\perp}$; BST extrapolation of the ratios $\ln(r'_L/r_{L-1})/\ln[L/(L-1)]$ yields $\nu_\perp=1.0962(7)$.

The moment ratios $r_{3111}\equiv m_3/m_1^3$ also exhibit crossings that converge to λ_c. More surprisingly, the product $m_{-1}m_1$ exhibits crossings and appears to approach a well-defined limit 1.366(1) as $L\to\infty$ at the critical point; simulations ($L=1000$ and 2000), yield 1.374(2) for this quantity. (Here $m_{-1}\equiv L\Sigma_C \bar{p}_C n_C^{-1}$ is the first negative moment of the particle density.) The reduced fourth cumulant, or kurtosis, given by $q(\lambda,L)=K_4/K_2^2$, where $K_2=m_2-m_1^2$ (the variance of the order parameter) and $K_4=m_4-4m_3m_1-3m_2^2+12m_2m_1^2-6m_1^4$, does not exhibit crossings but instead takes a pronounced minimum at a value $\lambda_{qm}(L)$ that converges to the critical value as $\lambda_c-\lambda_{qm}(L)\propto L^{-1.39(1)}$. (This property has been verified in simulations using $L=1000$ and 2000; departures from the minimum value are evident for $|\Delta|\geqslant 5\times 10^{-4}$ [25].) The sharpness of the minimum, as gauged by $q''=d^2q/d\lambda^2|_{\lambda_{qm}}$, appears to increase rapidly with size: $q''\propto L^{1.84(3)}$. (This is consistent with $q''\sim L^{2/\nu_\perp}$, as expected from FSS.) Since a negative kurtosis reflects a probability distribution that is broader at the maximum, and with shorter tails (compared to a Gaussian distribution with the same mean and variance), it is natural that q should be minimum at the critical point, where fluctuations are dominant. Precise estimates are also found for r_{3111} and q at the critical point: using BST extrapolation I find values of 1.5306(5) and −0.5015(5), compared with the simulation values of 1.526(3) and −0.505(3), respectively [24].

One may use the QS probability distribution to calculate the statistical entropy per site, $s=-L^{-1}\Sigma_j p_j \ln p_j$. In the large-$L$ limit, s should be zero for $\lambda<\lambda_c$, since the QSD is concentrated on a set of configurations with vanishing density. As $L\to\infty$, one expects $ds/d\lambda|_{\lambda_c}$ to diverge (as is the case for $d\rho/d\lambda$), and to attain a maximum at some $\lambda>\lambda_c$, approaching zero as $\lambda\to\infty$. Since the entropy cannot exceed that of a random mixture with the same density $0\leqslant s(\lambda) \leqslant s_0\equiv -\rho\ln\rho-(1-\rho)\ln(1-\rho)$. (In general one expects $s<s_0$, due to correlations between particle positions.) It is therefore of interest to analyze the reduced entropy $\tilde{s}(\lambda,L)\equiv s(\lambda,L)/s_0[\rho(\lambda,L)]$ (see Fig. 1). Extrapolation yields the estimate $\tilde{s}(\lambda_c)\to 0.68$ as $L\to\infty$. There is evidence that $\tilde{s}$ is singular at λ_c: the derivative $d\tilde{s}/d\lambda|_{\lambda_c}$ appears to grow linearly with the system size.

It is natural to inquire whether the QS probability distribution exhibits any simplifying features. In an equilibrium lattice gas with interactions that do not extend beyond nearest neighbors, for example, the probability of a configuration depends only on the number of particles N and nearest-neighbor pairs P. In the CP, by contrast, I find that the QS probability of each configuration in a given (N,P) class is distinct (the probabilities typically vary over an order of magnitude or more, even far from the critical point). In a broad sense, this is because, unlike in equilibrium, not all annihilation events possess a complementary creation event. For similar reasons, it does not appear likely that the QSD could be obtained via the maximization of the statistical entropy, subject to some simple set of constraints.

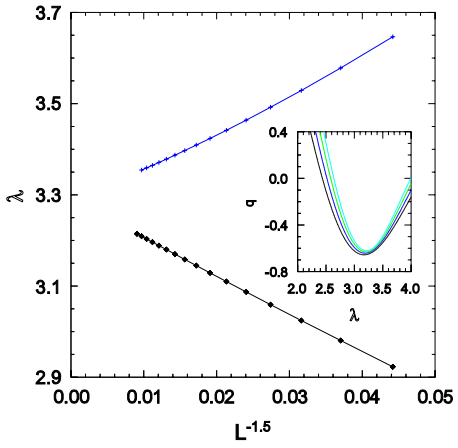

FIG. 2. (Color online) Contact process: values of λ at crossings of r_{211} (upper set) and at kurtosis minima (lower set) versus $1/L^{1.5}$. Inset: kurtosis q versus λ for (lower to upper) $L=17$, 19, 21, and 23.

To test the robustness of the approach developed above, I apply it to the pair contact process (PCP). In the PCP [27,28], each site is again either occupied or vacant, but all transitions involve a pair of particles occupying nearest-neighbor sites, called a pair in what follows. A pair annihilates itself at rate p, and with rate $1-p$ creates a new particle at a randomly chosen site neighboring the pair, if this site is vacant. Any configuration lacking a pair of nearest-neighbor occupied sites is absorbing. Thus the order parameter ρ is the density of nearest-neighbor occupied pairs. Simulation results [24,27,28] place the PCP in the same universality class as the CP (namely, that of directed percolation). Unlike the CP, for which quite precise results have been derived via series expansions, there are no reliable predictions from series or other analytic methods. Moreover, since the absorbing configuration is not unique, the diagonalization method of Ref. [8] is not applicable.

Using, as before, BST extrapolation of the estimates $p_{c,L}$ derived from crossings of R_L, I find $p_c=0.077091(7)$, consistent with the best available estimate of 0.077092(1) [29]. Estimates for z, $\nu_\perp$, and other critical parameters are listed in Table I; agreement with literature values is not quite as good as for the simpler CP. (In this case the estimates for $\nu_\perp$ and $\beta/\nu_\perp$ are obtained from the data for dr_A/dp and ρ at p_c, assuming the usual finite-size scaling forms, and estimating the associated critical exponents by extrapolating the associated local slopes [7].) Unlike the CP, the derivative of reduced entropy $d\tilde{s}/dp$ of the PCP appears to be finite at the critical point. This is likely due to the fact that the QS particle density in the critical PCP is nonzero, even in the infinite-size limit.

A further test of the method is its application to a model in the conserved directed percolation (CDP) universality class, which also includes the conserved stochastic sandpile [30,31]. In this model, a fixed number n of particles hop at a unit rate, without bias, to either nearest neighbor on a ring of L sites; the density $\zeta=n/L$ is conserved. There is no restriction on the number of particles occupying a given site; an

isolated particle, however, has a rate s to become inactive. An inactive particle cannot hop, and is only reactivated if another particle hops onto its site. Thus configurations with all particles inactive are absorbing. Probabilistic arguments, mean-field theory, and simulation show that this model exhibits a line of continuous phase transitions in the $\zeta - s$ plane, belonging to the CDP-stochastic sandpile class [32]. Here I study the case $\zeta = 1/2$; the rapid growth in the number of configurations ($\propto 2.8^L$) limits this preliminary study to $L \leqslant 16$.

Analysis of the ratios R_L (for $L = 8, 10, \ldots, 16$) yields $s_c = 0.1381(17)$. [Simulations [32] furnish $s_c = 0.1385(1)$.] The corresponding estimate for m_{211} at the critical point is 1.148(4). Analysis following the methods outlined above furnishes $\beta/\nu_\perp = 0.25(2)$, $z = 1.62(2)$, and $\nu_\perp = 1.35(9)$, while the literature values [33,34] for CDP are $m_{211} = 1.142(8)$, $\beta/\nu_\perp = 0.213(6)$, $z = 1.50(4)$, and $\nu_\perp = 1.34(2)$. In this case QS analysis provides reliable results for the location of the critical point and the ratio m_{211}, but yields inaccurate and/or imprecise estimates for critical exponents, probably due to the small sizes used in the present study. (Extension to another model in the CDP class, in which the number of configurations grows more slowly with L, is planned for future work.)

In summary, I show that analysis of exact (numerical) quasistationary properties on relatively small rings yields remarkably precise results for critical properties at an absorbing-state phase transition. The method can be applied to a broader range of models than are accessible via series expansion or exact diagonalization. Access to the full QSD permits analysis of the entropy and order-parameter moment ratios. Deriving the QSD involves rather modest programming and computational effort: the results reported here can be obtained in a few days on a fast microcomputer. The method provides a valuable check on simulations, may be useful in the study of metastable states, and may serve as the basis for phenomenological renormalization group approaches.

ACKNOWLEDGMENTS

I thank Robert Ziff for valuable suggestions. This work was supported by CNPq and FAPEMIG, Brazil.

[1] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[2] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1990).
[3] T. E. Harris, Ann. Probab. **2**, 969 (1974).
[4] T. Liggett, *Interacting Particle Systems* (Springer-Verlag, Berlin, 1985).
[5] R. Dickman, M. A. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000).
[6] T. Bohr, M. van Hecke, R. Mikkelsen, and M. Ipsen, Phys. Rev. Lett. **86**, 5482 (2001), and references therein.
[7] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1999).
[8] J. R. G. de Mendonça, J. Phys. A **32**, L467 (1999).
[9] J. Hooyberghs and C. Vanderzande, Phys. Rev. E **63**, 041109 (2001).
[10] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[11] G. Ódor, Rev. Mod. Phys. **76**, 663 (2004).
[12] S. Lübeck, Int. J. Mod. Phys. B **18**, 3977 (2004).
[13] K. A. Takeuchi, M. Kuroda, H. Chaté, and M. Sano, Phys. Rev. Lett. **99**, 234503 (2007).
[14] R. Dickman and R. Vidigal, J. Phys. A **35**, 1145 (2002).
[15] M. M. de Oliveira and R. Dickman, Phys. Rev. E **71**, 016129 (2005).
[16] R. Dickman and M. M. de Oliveira, Physica A **357**, 134 (2005); M. M. de Oliveira and R. Dickman, Braz. J. Phys. **36**, 685 (2006).
[17] F. Ginelli, H. Hinrichsen, R. Livi, D. Mukamel, and A. Torcini, J. Stat. Mech.: Theory Exp. (2006), P08008.
[18] R. Dickman, Phys. Rev. E **65**, 047701 (2002).
[19] I. Jensen and R. Dickman, J. Stat. Phys. **71**, 89 (1993).
[20] M. E. Fisher and M. N. Barber, Phys. Rev. Lett. **28**, 1516 (1972).
[21] *Finite-size Scaling and Numerical Simulations of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990).
[22] W. Press, S. Teukolsky, W. Vetterling, and B. Flannery, *Numerical Recipes* (Cambridge University Press, Cambridge, 1992).
[23] M. Henkel and G. Schütz, J. Phys. A **21**, 2617 (1988).
[24] R Dickman and J. Kamphorst Leal da Silva, Phys. Rev. E **58**, 4266 (1998).
[25] Detailed results on moment ratios will be reported elsewhere.
[26] I. Jensen, J. Phys. A **32**, 5233 (1999).
[27] I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993).
[28] I. Jensen and R. Dickman, Phys. Rev. E **48**, 1710 (1993).
[29] M. M. de Oliveira and R. Dickman, Phys. Rev. E **74**, 011124 (2006).
[30] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).
[31] S. S. Manna, J. Phys. A **24**, L363 (1991); J. Stat. Phys. **59**, 509 (1990).
[32] R. Dickman and V. Sidoravicius (unpublished).
[33] J. J. Ramasco, M. A. Muñoz, and C. A. da Silva Santos, Phys. Rev. E **69**, 045105(R) (2004).
[34] R. Dickman, Phys. Rev. E **73**, 036131 (2006).